2025.10.28. 积分的计算.

0. 请大家复习书中含 $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$, x^2+a^2, $x-a^2$ 的积分（多为三角换元）. 无论在分子还是分母.

另外，一定背下来 $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $(\arctan x)' = \frac{1}{1+x^2}$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C, \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan x + C.$$

☆记住 $\int \frac{Bx+C}{x^2+px+q}dx$ $(p^2-4q<0$ 时$)$.

$$= \int \frac{\frac{B}{2}\cdot(2x+p)+(C-\frac{Bp}{2})}{x^2+px+q}dx = \frac{B}{2}\int \frac{2x+p}{x^2+px+q}dx + (C-\frac{Bp}{2})\int \frac{dx}{x^2+px+q}$$

$$= \frac{B}{2}\int \frac{d(x^2+px+q)}{x^2+px+q} + \left(C-\frac{Bp}{2}\right)\int \frac{dx}{(x+\frac{p}{2})^2+(q-p^2/4)}$$

$$= \frac{B}{2}\ln(x^2+px+q) + \frac{C-\frac{Bp}{2}}{\sqrt{q-\frac{p^2}{4}}}\arctan\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + \tilde{C}.$$

记住这个过程而不是结果.

(☆) 如果需处理 $\int \frac{dx}{(x^2+a^2)^n}$ 型积分.

$$I_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} - \int x\cdot d\frac{1}{(x^2+a^2)^n}$$

$$= \frac{x}{(x^2+a^2)^n} - (-n)\int \frac{x\cdot 2x}{(x^2+a^2)^{n+1}}dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\cdot\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$$

$$\frac{x^2}{(x^2+a^2)^{n+1}} = \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}} = \frac{1}{(x^2+a^2)^n} - a^2\frac{1}{(x^2+a^2)^{n+1}}$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{dx}{(x^2+a^2)^n} - 2na^2\int \frac{dx}{(x^2+a^2)^{n+1}}$$

$$\Rightarrow I_{n+1} = \frac{2n-1}{2na^2}I_n + \frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n}$$ 递推求解.

1. 其它递推积分.

① $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

$$I_n = \int \sin^n x\,dx = -\int \sin^{n-1}x\,d\cos x = -\sin^{n-1}x\cos x + \int \cos x\,d\sin^{n-1}x$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \underbrace{\cos^2 x}_{1-\sin^2 x}\sin^{n-2}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\underbrace{\int \sin^{n-2}x\,dx}_{I_{n-2}} - (n-1)\underbrace{\int \sin^n x\,dx}_{I_n}$$

$$\Rightarrow I_n = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}I_{n-2}.$$

特别地, $\cos x\sin^{n-1}x\Big|_0^{\pi/2} = 0$.

故记 $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ 为 I_n. 有 $I_n = \frac{n-1}{n}I_{n-2}$.

因为 $I_0 = \frac{\pi}{2}$, $I_1 = 1$.

故 $I_{2n+1} = \frac{2n\times(2n-2)\times\cdots\times 2}{(2n+1)\times(2n-1)\cdots\times 3}I_1 = \frac{(2n)!!}{(2n+1)!!}$

$$I_{2n} = \frac{(2n-1)\times(2n-3)\times\cdots\times 1}{2n\times(2n-2)\times\cdots\times 2}I_0 = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}.$$

② $\int_0^1 x^m(\ln x)^n\,dx$.

记 $I(m,n) = \int_0^1 x^m(\ln x)^n\,dx = \frac{1}{m+1}\int_0^1 (\ln x)^n\,dx^{m+1}$

$$= \underbrace{\frac{x^{m+1}}{m+1}(\ln x)^n\Big|_0^1}_{=0} - \frac{1}{m+1}\int_0^1 x^{m+1}\cdot n(\ln x)^{n-1}\cdot\frac{1}{x}\,dx$$

$$= -\frac{n}{m+1}\cdot\int_0^1 x^m(\ln x)^{n-1}\,dx = -\frac{n}{m+1}I(m,n-1)$$

递推: $I(m,n) = -\frac{n}{m+1}I(m,n-1) = \frac{n(n-1)}{(m+1)^2}I(m,n-2)$

$$= \cdots = \frac{(-1)^n n!}{(m+1)^n}I(m,0)$$

$I(m,0) = \int_0^1 x^m dx = \frac{1}{m+1}$ $\Rightarrow I(m,n) = (-1)^n\frac{n!}{(m+1)^{n+1}}$.

2. 陷阱定积分. (难). 仅说明注意原函数可能间断.

(1) 计算 $f(x)=\arctan\frac{3x(x^2-1)}{x^4-4x^2+1}$ 导数

(2) 计算 $\int_0^1 \frac{1+x^4}{1+x^6}dx$.

(1) $f'(x)=\frac{1+x^4}{1+x^6}$.

(2) 注意到直接代入 $f(x)\Big|_0^1=0$ 肯定不对.

因为 $\frac{1+x^4}{1+x^6}>0$!!!

问题出在 $x=\frac{\sqrt{3}-1}{\sqrt{2}}$ 处, $f(x)$ 间断

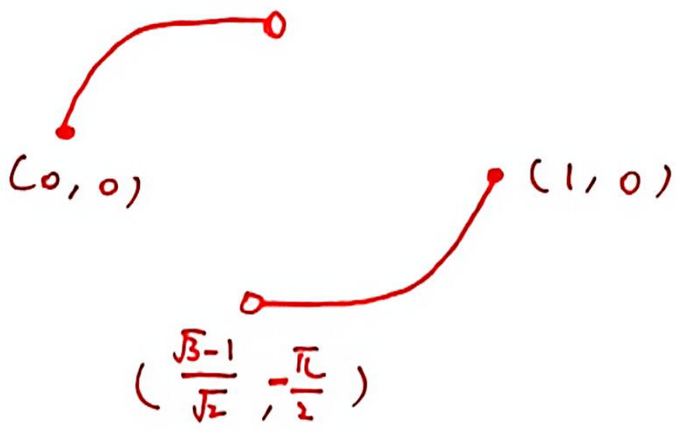
$\lim\limits_{x\to\frac{\sqrt{3}-1}{\sqrt{2}}^-} f(x)=\arctan(+\infty)=\frac{\pi}{2}$

$\lim\limits_{x\to\frac{\sqrt{3}-1}{\sqrt{2}}^+} f(x)=\arctan(-\infty)=-\frac{\pi}{2}$.

连续的原函数应为 $\tilde{f}(x)=\begin{cases} f(x), & 0\le x\le\frac{\sqrt{3}-1}{\sqrt{2}} \\ f(x)+\pi, & \frac{\sqrt{3}-1}{\sqrt{2}}<x\le 1 \end{cases}$

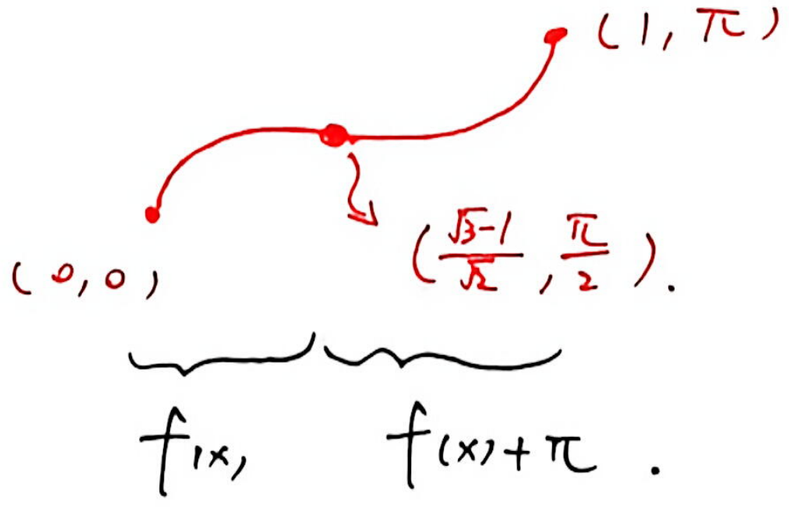
故 $\int_0^1 \frac{1+x^4}{1+x^6}dx=\tilde{f}(x)\Big|_0^1=\pi$.

图像. $f(x)$ 形如



这不符合 N-L 公式条件

一个连续的原函数 $\tilde{f}(x)$ 形如



这符合 N-L 公式条件.

3. 有理型定积分.

① $\int \frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}\,dx$

部分分式 $\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)} = \frac{A}{2x-1} + \frac{B}{2x+3} + \frac{C}{2x-5}$.

计算 A, B, C.

$$A + (2x-1)\left(\frac{B}{2x+3} + \frac{C}{2x-5}\right) = \frac{4x^2+4x-11}{(2x+3)(2x-5)}$$

代入 $x=\frac{1}{2}$

$$A = \left.\frac{4x^2+4x-11}{(2x+3)(2x-5)}\right|_{x=\frac{1}{2}} = \frac{1}{2}$$

同理. $B = \left.\frac{4x^2+4x-11}{(2x-1)(2x-5)}\right|_{x=-3/2} = -\frac{1}{4}$

$$C = \left.\frac{4x^2+4x-11}{(2x-1)(2x+3)}\right|_{x=5/2} = \frac{3}{4}.$$

$$\Rightarrow \text{原式} = \int \frac{1}{2}\cdot\frac{1}{2x-1} - \frac{1}{4}\cdot\frac{1}{2x+3} + \frac{3}{4}\cdot\frac{1}{2x-5}\,dx$$

$$= \frac{1}{4}\ln|2x-1| - \frac{1}{8}\ln|2x+3| + \frac{3}{8}\ln|2x-5| + C.$$

② $\int \frac{dx}{(x+2)(x^2+1)}$ 部分分式 $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$.

计算 A. $A + \frac{Bx+C}{x^2+1}(x+2) = \frac{1}{x^2+1}$ 令 $x=-2$. $A = \frac{1}{5}$.

计算 B, C. $Bx+C + \frac{x^2+1}{5(x+2)} = \frac{1}{x+2}$ 令 $x=0$. $\Rightarrow C+\frac{1}{10}=\frac{1}{2}$, $C=\frac{2}{5}$

令 $x=4$. $\Rightarrow B+\frac{2}{5}+\frac{2}{15}=\frac{1}{3}$

$B=-\frac{1}{5}$

$$\Rightarrow \text{原式} = \int \frac{1}{5}\cdot\frac{1}{x+2} + \frac{-x+2}{5(x^2+1)}\,dx$$

$$\int \frac{dx}{x+2} = \ln|x+2| + C.$$

$$\int \frac{-x+2}{x^2+1} = -\frac{1}{2}\ln|x^2+1| + 2\arctan x + C$$

$$\Rightarrow \text{原式} = \frac{1}{5}\ln|x+2| - \frac{1}{10}\ln(x^2+1) + \frac{2}{5}\arctan x + C.$$

③ $$\int_0^{\pi/2} \frac{\sin x\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{\sin x}{1+\sin^2 x}\,d\sin x$$

$$= \frac{1}{2}\ln(1+\sin^2 x)\Big|_{\sin x=0}^{\sin x=1} = \frac{1}{2}\ln 2.$$

4. 分式变换.

启示：谁丑换元谁.

① $$\int \frac{dx}{\sqrt[3]{(x+1)(x-1)^5}} = \int \sqrt[3]{\frac{x-1}{x+1}}\cdot\frac{dx}{(x-1)^2}$$

令 $t = \sqrt[3]{\frac{x-1}{x+1}}$ $\quad t^3 = \frac{x-1}{x+1} = 1-\frac{2}{x+1}$

$x = \frac{2}{1-t^3} - 1$ $\quad x+1 = \frac{2}{1-t^3}$

$dx = \frac{6t^2}{(1-t^3)^2}dt$ $\quad x-1 = \frac{2t^3}{1-t^3}$

$$= \int t\cdot\frac{(1-t^3)^2}{4t^6}\cdot\frac{6t^2}{(1-t^3)^2}dt$$

$$= \frac{3}{2}\int\frac{dt}{t^3} = -\frac{3}{4t^2}+C = -\frac{3}{4}\left(\frac{x+1}{x-1}\right)^{2/3}+C$$

② $$\int \frac{dx}{\sqrt[3]{(2+x)(2-x)^5}} = \int \sqrt[3]{\frac{2-x}{2+x}}\cdot\frac{dx}{(2-x)^2}.$$

令 $t = \sqrt[3]{\frac{2-x}{2+x}}$ $\quad t^3 = \frac{2-x}{2+x} = \frac{4}{x+2} - 1.$

$x+2 = \frac{4}{t^3+1}$ $\quad x = \frac{4}{t^3+1} - 2$

$2-x = \frac{4t^3}{t^3+1}$ $\quad dx = \frac{-4\times 3t^2}{(t^3+1)^2}dt = -\frac{12t^2dt}{(t^3+1)^2}$

$$= \int t\cdot\frac{(t^3+1)^2}{16t^6}\cdot\left(-\frac{12t^2}{(t^3+1)^2}\right)dt = -\frac{3}{4}\int\frac{dt}{t^3}$$

$$= +\frac{3}{8}\cdot\frac{1}{t^2}+C = \frac{3}{8}\left(\frac{2+x}{2-x}\right)^{2/3}+C.$$

5. 特殊性质.

① $\int_{-1}^{1} \frac{x^2(1+\arcsin x)}{1+x^2} dx = \int_{-1}^{1} \frac{x^2}{1+x^2} dx$ (因为 $\frac{x^2}{1+x^2}\arcsin x$ 奇)

$= (x - \arctan x)\Big|_{-1}^{1} = 2 - \frac{\pi}{2}$.

② $\int_0^{2\pi} \sqrt{1-\sin 2x}\, dx = \int_0^{2\pi} \sqrt{(\sin x - \cos x)^2}\, dx$

$= \int_0^{2\pi} |\sin x - \cos x|\, dx$

$= \sqrt{2}\int_0^{2\pi} |\sin(x-\frac{\pi}{4})|\, dx$ ↘ $|\sin(x-\frac{\pi}{4})|$ 周期为 π 导致两个.

$= \sqrt{2}\int_0^{2\pi} |\sin x|\, dx$ ↘ 周期为 π

$= 2\sqrt{2}\int_0^{\pi} |\sin x|\, dx$

$= 2\sqrt{2}\int_0^{\pi} \sin x\, dx = 4\sqrt{2}$.

6. XYT 讲义最后几个补充题.

① $f'(2+\cos x) = \tan^2 x + \sin^2 x$. 问 $f(x) = ?$

$f(2+\cos x) = \int (\tan^2 x + \sin^2 x)\, d(2+\cos x)$

$= \int (\tan^2 x + \sin^2 x)\, d\cos x$. → 令 $t = \cos x$

$\tan^2 x = \frac{1}{t^2} - 1 = \frac{1-t^2}{t^2}$

$\sin^2 x = 1 - t^2$

$= \int (1-t^2)(\frac{1}{t^2}+1)\, dt$

$= \int \frac{1-t^4}{t^2}\, dt = -\frac{1}{t} - \frac{t^3}{3} + C$ 奇

i.e. $f(t+2) = -\frac{1}{t} - \frac{t^3}{3} + C$. 故. $f(x) = -\frac{1}{(x-2)} - \frac{1}{3}(x-2)^3 + C$.

② 分母有理化技术.

$$\int \frac{\sqrt{x(x+1)}}{\sqrt{x}+\sqrt{x+1}}\,dx = \int \sqrt{x(x+1)}\,(\sqrt{x+1}-\sqrt{x})\,dx$$

$$= \int \sqrt{x}(x+1)\,dx - \int x\sqrt{x+1}\,dx$$

(i) $\int \sqrt{x}(x+1)\,dx = \int x^{3/2}+x^{1/2}\,dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$

(ii) $\int x\sqrt{x+1}\,dx \xrightarrow{\text{换 } y=x+1} \int (y-1)\sqrt{y}\,dy$

$$= \int y^{3/2} - y^{1/2}\,dy = \frac{2}{5}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

$\Rightarrow$ 原式 $= \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} - \frac{2}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C$.

③ $\int \frac{x\ln x}{(1+x^2)^2}\,dx = \frac{1}{2}\int \frac{\ln x}{(1+x^2)^2}\,dx^2 = -\frac{1}{2}\int \ln x\, d\frac{1}{1+x^2}$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int \frac{1}{1+x^2}\,d\ln x$$

$$\int \frac{d\ln x}{1+x^2} = \int \frac{dx}{x(1+x^2)} = \frac{1}{2}\int \frac{dx^2}{x^2(1+x^2)} = \frac{1}{2}\ln\frac{x^2}{1+x^2} + C.$$

$\Rightarrow$ 原式 $= -\frac{\ln x}{2(1+x^2)} + \frac{1}{4}\ln\frac{x^2}{1+x^2} + C$

④ $\int e^x \sin x\,dx$. (启示：e^x做分部积分总不变，$\sin x/\cos x$ 总周期变化 敢于做分部积分，必然最后得到一个方程).

Step1. 回忆 $\int e^x \sin x\,dx$ 的计算.

方法1. $\int e^x \sin x\,dx = \int \sin x\,de^x = e^x\sin x - \int e^x\cos x\,dx$

$$\int e^x\cos x\,dx = \int \cos x\,de^x = e^x\cos x + \int e^x\sin x\,dx$$

$$\Rightarrow \int e^x\sin x\,dx = e^x(\sin x - \cos x) - \int e^x\sin x\,dx$$

$$\Rightarrow \int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

方法2 凑配对偶式 $I=\int e^x\sin x\,dx$, $J=\int e^x\cos x\,dx$.

$I+J=\int e^x(\sin x+\cos x)\,dx = e^x\sin x+C$ （$\cos x=(\sin x)'$）

$J-I=\int e^x(\cos x-\sin x)\,dx = e^x\cos x+C$. （$\cos x=(\cos x)'$）

$\Rightarrow \int e^x\sin x\,dx = I = \frac{1}{2}e^x(\sin x-\cos x)+C$.

（反问题）$\int e^x x\sin x\,dx = \int x\sin x\,de^x = e^x x\sin x-\int e^x(\sin x+x\cos x)\,dx$

$= e^x x\sin x-\int e^x\sin x\,dx-\int e^x x\cos x\,dx$

$= e^x x\sin x-\frac{1}{2}e^x(\sin x-\cos x)-\int e^x x\cdot\cos x\,dx$.

$\int e^x x\cos x\,dx = \int x\cos x\,de^x = e^x x\cos x-\int e^x(\cos x-x\sin x)\,dx$

$= e^x x\cos x-\int e^x\cos x\,dx+\int e^x x\sin x\,dx$

$= e^x x\cos x-\frac{1}{2}e^x(\sin x+\cos x)+\int e^x x\sin x\,dx$

故 $\int e^x x\sin x\,dx = e^x x(\sin x-\cos x)+e^x\cos x-\int e^x x\sin x\,dx$

$\Rightarrow \int e^x x\sin x\,dx = \frac{1}{2}e^x(x\sin x-x\cos x+\cos x)+C$.

⑤ $\int \frac{1+x}{x(1+xe^x)}\,dx$. 分析：无法处理 xe^x. 两种手段

(i) 换元 $t=xe^x$ 反函数难写 故不好做

(ii) 凑微分 $d\,xe^x$ 主动尝试配以下形式。 $d\,xe^x = e^x(x+1)\,dx$.

$= \int \frac{e^x(x+1)}{xe^x(1+xe^x)}\,dx$

$= \int \frac{d\,xe^x}{xe^x(1+xe^x)} = \int \frac{1}{xe^x}-\frac{1}{1+xe^x}\,d\,xe^x$

$= \ln\left|\frac{xe^x}{1+xe^x}\right|+C$.

⑥ $\int \sqrt{\tan x}\,dx$ 仍然将无法处理的 $\sqrt{\tan x}$ 直接换元.

令 $t=\sqrt{\tan x}$. $x=\arctan t^2$.

$dx=\frac{2t}{1+t^4}dt$

$=\int \frac{2t^2}{1+t^4}dt$.

方法1. (有理函数积分).

注意到 $t^4+1=(t^2+1)^2-2t^2=(t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)$

故部分分式 $\frac{2t^2}{1+t^4}=\frac{At+B}{t^2-\sqrt{2}t+1}+\frac{Ct+D}{t^2+\sqrt{2}t+1}$.

(i) 考虑分子 t^3 项系数文. $0=A+C$ 故 $C=-A$.

简化为 $\frac{2t^2}{1+t^4}=\frac{At+B}{t^2-\sqrt{2}t+1}-\frac{At+D}{t^2+\sqrt{2}t+1}$

(ii) 考虑分子常数项. $0=B-D$. 故 $B=D$.

简化为. $\frac{2t^2}{1+t^4}=\frac{At+B}{t^2-\sqrt{2}t+1}-\frac{At+B}{t^2+\sqrt{2}t+1}$

(iii) 考虑分子 t 项系数. $0=A-A+\sqrt{2}B-(-\sqrt{2})B$

$=2\sqrt{2}B$. 故 $B=0$.

简化为 $\frac{2t^2}{1+t^4}=\frac{At}{t^2-\sqrt{2}t+1}-\frac{At}{t^2+\sqrt{2}t+1}$.

(iv) 令 $t=\sqrt{2}$. $\frac{2\times 2}{1+4}=\frac{\sqrt{2}A}{2-2+1}-\frac{\sqrt{2}A}{2+2+1}$

即 $\frac{4}{5}=\sqrt{2}A\times(1-\frac{1}{5})=\frac{4}{5}\sqrt{2}A$ $\Rightarrow A=\frac{\sqrt{2}}{2}$.

综上: $\frac{2t^2}{1+t^4}=\left(\frac{t}{t^2-\sqrt{2}t+1}-\frac{t}{t^2+\sqrt{2}t+1}\right)\cdot\frac{\sqrt{2}}{2}$

接下来套用 $\int\frac{Bx+C}{x^2+px+q}dx=\frac{B}{2}\ln(x^2+px+q)+\frac{C-\frac{Bp}{2}}{\sqrt{q-p^2/4}}\arctan\frac{x+\frac{p}{2}}{\sqrt{q-p^2/4}}+$ 常数.

$\frac{\sqrt{2}}{2}\int\frac{t}{t^2-\sqrt{2}t+1}dt=\frac{\sqrt{2}}{4}\ln(t^2-\sqrt{2}t+1)+\frac{\sqrt{2}}{2}\arctan(\sqrt{2}t-1)+C$

$-\frac{\sqrt{2}}{2}\int\frac{t}{t^2+\sqrt{2}t+1}dt=\frac{\sqrt{2}}{4}\ln(t^2+\sqrt{2}t+1)-\frac{\sqrt{2}}{2}\arctan(\sqrt{2}t+1)+C$.

故 $\int \sqrt{\tan x}\,dx = \int \frac{2t^2}{1+t^4}\,dt$

$= \frac{\sqrt{2}}{4}\ln\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1} + \frac{\sqrt{2}}{2}\left(\arctan(\sqrt{2}t-1)+\arctan(\sqrt{2}t+1)\right)$

$+C$.

这里用 $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\Rightarrow \arctan X + \arctan Y = \arctan\frac{X+Y}{1-XY}$.

故 $\arctan(\sqrt{2}t-1)+\arctan(\sqrt{2}t+1) = \arctan\frac{2\sqrt{2}t}{1-(2t^2-1)}$ ~~$=$~~ $\Longrightarrow$

$= \arctan\frac{\sqrt{2}t}{1-t^2}$.

$\Rightarrow \int\sqrt{\tan x}\,dx = \frac{\sqrt{2}}{4}\ln\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1} + \frac{\sqrt{2}}{2}\arctan\frac{\sqrt{2}t}{1-t^2} + C$

$= \frac{\sqrt{2}}{4}\ln\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1} + \frac{\sqrt{2}}{2}\arctan\frac{\sqrt{2\tan x}}{1-\tan x} + C$.

↳ $\arctan X$

$= \arctan\left(-\frac{1}{X}\right)+\frac{\pi}{2}$

所以结果是一样的.

方法2. $I = \int\frac{2t^2}{1+t^4}\,dt$ 再记对偶式 $J = \int\frac{dt}{1+t^4}$

$I+J = \int\frac{t^2+1}{1+t^4}\,dt = \int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt = \int\frac{d(t-\frac{1}{t})}{(t-\frac{1}{t})^2+2}$

$= \frac{1}{\sqrt{2}}\arctan\frac{t-\frac{1}{t}}{\sqrt{2}} + C. = \frac{\sqrt{2}}{2}\arctan\frac{t^2-1}{\sqrt{2}t} + C$

$I-J = \int\frac{t^2-1}{t^4+1}\,dt = \int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt = \int\frac{d(t+\frac{1}{t})}{(t+\frac{1}{t})^2-2}$

$= \frac{1}{2\sqrt{2}}\ln\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|$

$= \frac{\sqrt{2}}{4}\ln\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1} + C$

故原式 $\int\sqrt{\tan x}\,dx = \int\frac{2t^2}{1+t^4}\,dt = 2I = \frac{\sqrt{2}}{4}\ln\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1} + \frac{\sqrt{2}}{2}\arctan\frac{t^2-1}{\sqrt{2}t} + C$

$= \frac{\sqrt{2}}{4}\ln\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1} + \frac{\sqrt{2}}{2}\arctan\frac{\tan x-1}{\sqrt{2\tan x}} + C$.

补. 二项式微分式积分的 Chebyshev 定理.

Recall. $\int \frac{dx}{\sqrt[3]{(x+1)(x-1)^5}} = \int \sqrt[3]{\frac{x-1}{x+1}} \frac{dx}{(x-1)^2}$

通过换元 $t = \sqrt[3]{\frac{x-1}{x+1}}$ 解决.

问题. 将 5 改成其它数字可以吗?

以 3, 7 为例.

① $\int \frac{dx}{\sqrt[3]{x+1}\,(x-1)}$ 换元 $t = \sqrt[3]{x+1}$.

$x = t^3 - 1$, $x-1 = t^3 - 2$

$dx = 3t^2 dt$

$= \int \frac{3t^2 dt}{t(t^3-2)} = 3\int \frac{t\,dt}{t^3-2}$ 可以写为部分分式.

② $\int \frac{dx}{\sqrt[3]{(x+1)(x-1)^7}}$ 则无法写出原函数!!!

一般地, 有 Chebyshev 定理.

对于形如 $\int x^p (a+bx)^q dx$ 型积分.

当且仅当 $p, q, p+q$ 中有整数时, 是初等函数.

特别地, 换元 $t = x+1$

原式 $= \int t^{-\frac{1}{3}} (t-2)^{-\frac{7}{3}} dt$. $-\frac{1}{3}, -\frac{7}{3}, -\frac{8}{3}$ 都不是整数

因此积不出来.