2026.6.9. Fourier级数.

0. 常用积分回顾.

① $\int_0^{\varphi} \sin nx\,dx \xrightarrow{t=nx} \frac{1}{n}\int_0^{n\varphi} \sin t\,dt = \frac{1}{n}\cos t\Big|_{n\varphi}^{0}$

$= \frac{1}{n}(1-\cos n\varphi)$

$\int_0^{\varphi} \cos nx\,dx \xrightarrow{t=nx} \frac{1}{n}\int_0^{n\varphi} \cos t\,dt = \frac{\sin t}{n}\Big|_0^{n\varphi} = \frac{\sin n\varphi}{n}$

② $\int_0^{\pi} x\sin nx\,dx \xrightarrow{t=nx} \frac{1}{n^2}\int_0^{n\pi} t\sin t\,dt.$

$\int t\sin t\,dt = -\int t\,d\cos t = -t\cos t + \int \cos t\,dt$

$= \sin t - t\cos t.$

原积分 $= \frac{1}{n^2}(\sin t - t\cos t)\Big|_0^{n\pi} = \frac{1}{n^2}[(0-(-1)^n n\pi)-0]$

$= \frac{(-1)^{n+1} n\pi}{n^2} = \frac{(-1)^{n+1}\pi}{n}.$

$\int_0^{\pi} x\cos nx\,dx \xrightarrow{t=nx} \frac{1}{n^2}\int_0^{n\pi} t\cos t\,dt$

$\int t\cos t\,dt = \int t\,d\sin t = t\sin t - \int \sin t\,dt = t\sin t + \cos t.$

原积分 $= \frac{1}{n^2}(t\sin t+\cos t)\Big|_0^{n\pi} = \frac{1}{n^2}[(-1)^n - 1] = \begin{cases} 0, & 2\mid n \\ -\frac{2}{n^2}, & 2\nmid n. \end{cases}$

③ $\int_0^{\pi} x^2\sin nx\,dx \xrightarrow{t=nx} \frac{1}{n^3}\int_0^{n\pi} t^2\sin t\,dt.$

$\int t^2\sin t\,dt = -\int t^2\,d\cos t = -t^2\cos t + \int 2t\cos t\,dt$

$= -t^2\cos t + 2\int t\,d\sin t = -t^2\cos t + 2t\sin t - 2\int \sin t\,dt$

$= -t^2\cos t + 2t\sin t + 2\cos t.$

原积分 $= \frac{1}{n^3}(-t^2\cos t + 2t\sin t + 2\cos t)\Big|_0^{n\pi}$

$= \frac{1}{n^3}[(-n^2\pi^2\cdot(-1)^n + 0 + 2(-1)^n) - (0+0+2)]$

$= \frac{(-1)^{n+1}}{n}\pi^2 + \frac{2}{n^3}[(-1)^n - 1].$

$$\int_0^{\pi} x^2\cos nx\,dx \xrightarrow{t=nx} \frac{1}{n^3}\int_0^{n\pi} t^2\cos t\,dt.$$

$$\int t^2\cos t\,dt = \int t^2 d\sin t = t^2\sin t - \int \sin t\cdot 2t\,dt$$

$$= t^2\sin t + 2\int t\,d\cos t = t^2\sin t + 2t\cos t - 2\int \cos t\,dt$$

$$= t^2\sin t + 2t\cos t - 2\sin t$$

原积分 $= \frac{1}{n^3}\left[\left(t^2\sin t + 2t\cos t - 2\sin t\right)\Big|_0^{n\pi}\right]$.

$$= \frac{1}{n^3}\left[\left(0 + 2n\pi(-1)^n - 0\right) - \left(0+0-0\right)\right]$$

$$= \frac{2\pi}{n^2}(-1)^n.$$

④ $\int e^{\alpha x}\sin\beta x\,dx$, $\int e^{\alpha x}\cos\beta x\,dx$.

注意到 $(e^{\alpha x}\sin\beta x)' = e^{\alpha x}(\alpha\sin\beta x + \beta\cos\beta x)$

$(e^{\alpha x}\cos\beta x)' = e^{\alpha x}(-\beta\sin\beta x + \alpha\cos\beta x)$.

那么, $(Ae^{\alpha x}\sin\beta x + Be^{\alpha x}\cos\beta x)'$

$$= e^{\alpha x}\cdot\left[(A\alpha - B\beta)\sin\beta x + (A\beta + B\alpha)\cos\beta x\right].$$

(i) 令 $\begin{cases} \alpha\cdot A - \beta\cdot B = 1 \\ \beta\cdot A + \alpha\cdot B = 0. \end{cases}$ 即 $\begin{cases} A = \frac{\alpha}{\alpha^2+\beta^2} \\ B = \frac{-\beta}{\alpha^2+\beta^2}. \end{cases}$

即 $\int e^{\alpha x}\sin\beta x\,dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x - \beta\cos\beta x)$

(ii) 令 $\begin{cases} \alpha\cdot A - \beta\cdot B = 0 \\ \beta\cdot A + \alpha\cdot B = 1. \end{cases}$ 即 $\begin{cases} A = \frac{\beta}{\alpha^2+\beta^2} \\ B = \frac{\alpha}{\alpha^2+\beta^2} \end{cases}$

即 $\int e^{\alpha x}\cos\beta x\,dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}(\beta\sin\beta x + \alpha\cos\beta x)$.

「关于 Fourier 级数的和函数」.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n \cos nx + b_n \sin nx) \triangleq S(x)$$

① Dirichlet 定理.

$f(x)$ 在 $[-\pi, \pi]$ 上分段连续、分段单调.

则 $S(x) = \begin{cases} f(x), & x \text{ 是连续点}, \\ \frac{1}{2}(f(x^+) + f(x^-)), & x \text{ 是间断点}. \end{cases}$

② $f(x)$ 在 $[-\pi, \pi]$ 上分段可微，收敛性同上.

Rmk. 根据以上判断 $S(x)$，并不需求出 a_n, b_n.

1. 求 Fourier 系数.

① $f(x) = x. \quad x \in [-\pi, \pi]$.

由奇性，$a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2(-1)^{n-1}}{n}.$$

$$\Rightarrow \quad x \sim 2\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \sin nx = \begin{cases} x, & x \in (-\pi, \pi) \\ 0, & x = \pm\pi \end{cases}$$

② $f(x) = |x|. \quad x \in [-\pi, \pi]$.

由偶性，$b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \, dx = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \begin{cases} 0, & 2 \mid n \\ -\frac{4}{n^2\pi}, & 2 \nmid n. \end{cases}$$

$$\Rightarrow |x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{\cos(2n-1)x}{(2n-1)^2} = |x|. \quad x \in [-\pi, \pi].$$

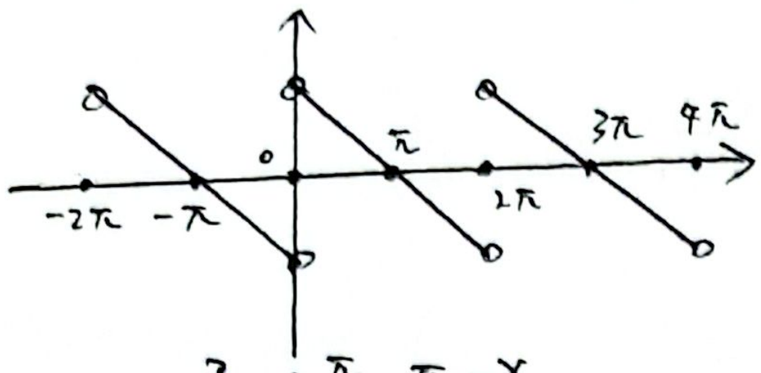
令 $x = \pi$. $\Rightarrow \quad \pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Rmk. 以上两例也可看作 $f(x) = x, \ x \in [0, \pi]$ 的奇延拓/正弦展开 和 偶延拓/余弦展开.

$$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + 2\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \sin nx. \quad x \in [0, \pi).$$

③ $f(x) = \frac{\pi - x}{2}$. $x \in [0, 2\pi]$.

这也可看作 $f(x) = \frac{\pi - x}{2}$. $x \in [0, \pi]$ 做奇延拓



故 $a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{\pi - x}{2}\sin nx\,dx = \int_0^{\pi}\sin nx\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{n}(1-(-1)^n) - \frac{1}{\pi}\cdot\frac{(-1)^{n+1}}{n}\pi = \frac{1}{n} - \frac{(-1)^n}{n} + \frac{(-1)^n}{n} = \frac{1}{n}.$$

$$\Rightarrow \frac{\pi - x}{2} \sim \sum_{n=1}^{+\infty}\frac{\sin nx}{n} = \begin{cases} \frac{\pi - x}{2}, & x \in (0, 2\pi) \\ 0, & x = 0, 2\pi. \end{cases}$$

这个可以记住!

Rmk. 答案并不是 $\frac{\pi}{2}$ 减去 ①② 中任何一个, 不要加减 Fourier 级数. 每个都好好算!

④ $f(x) = \frac{x^2}{4} - \frac{\pi}{2}x$. $x \in [0, 2\pi]$.

注意到 $f(0) = f(2\pi)$. 以及又对称轴 $x = \pi$.

这可看作 $f(x) = \frac{x^2}{4} - \frac{\pi}{2}x$. $x \in [0, \pi]$ 做偶延拓.

$b_n = 0$. $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = -\frac{\pi^2}{3}$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{1}{2\pi}\int_0^{\pi} x^2\cos nx\,dx - \int_0^{\pi} x\cos nx\,dx$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi}{n^2}(-1)^n - \frac{1}{n^2}((-1)^n - 1) = \frac{1}{n^2}.$$

$$\Rightarrow \frac{x^2}{4} - \frac{\pi}{2}x = -\frac{\pi^2}{6} + \sum_{n=1}^{+\infty}\frac{\cos nx}{n^2}. \quad x \in [0, 2\pi]$$

令 $x = 0$. 得 $\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

注2. $\frac{\pi - x}{2} = \sum_{n=1}^{+\infty} \frac{\sin nx}{n}$, $x \in (-\pi, \pi)$ 内闭一致收敛.

因此可以逐项求积分.

$$\Rightarrow \frac{\pi}{2}x - \frac{x^2}{4} = C - \sum_{n=1}^{+\infty} \frac{\cos nx}{n^2}$$

$$\Rightarrow \frac{x^2}{4} - \frac{\pi}{2}x = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \frac{\cos nx}{n^2} \quad . \quad = -\frac{\pi^2}{6} + \sum_{n=1}^{+\infty} \frac{\cos nx}{n^2} \quad .$$

⑤ $f(x) = \begin{cases} 1, & |x| \le \varphi \\ 0, & |x| > \varphi. \end{cases}$ $\varphi \in (0, \pi)$.

$|x| \le \pi$.

$b_n = 0$ (偶性).

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2\varphi}{\pi}$.

$a_n = \frac{2}{\pi}\int_0^{\varphi} \cos nx\,dx = \frac{2\sin n\varphi}{n\pi}$

$\Rightarrow f(x) \sim \frac{\varphi}{\pi} + \frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{\sin n\varphi}{n} \cdot \cos nx$.

$$= \begin{cases} 1, & |x| < \varphi \\ \frac{1}{2}, & |x| = \varphi \\ 0, & \varphi < |x| \le \pi \end{cases}.$$

令 $x = 0$. 得 $1 = \frac{\varphi}{\pi} + \frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{\sin n\varphi}{n}$

即 $\sum_{n=1}^{+\infty} \frac{\sin n\varphi}{n} = \frac{\pi}{2}\cdot\left(1 - \frac{\varphi}{\pi}\right) = \frac{\pi - \varphi}{2}$. $\varphi \in (0, \pi)$.

⑧ $f(x) = \begin{cases} 1, & 0 \le x \le \varphi \\ -1, & -\varphi \le x < 0 \\ 0, & \varphi < |x| \le \pi. \end{cases}$ $a_n = 0$ (奇性).

$b_n = \frac{2}{\pi}\int_0^{\varphi} \sin nx\,dx = \frac{2}{\pi}\cdot\frac{1}{n}(1 - \cos n\varphi)$.

$\Rightarrow f(x) \sim \frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{1-\cos n\varphi}{n} \cdot \sin nx$. $= \begin{cases} 1, & 0 < x \le \varphi \\ -1, & -\varphi < x < 0 \\ 0, & x = 0 \text{ 或 } \varphi < |x| \le \pi \\ \frac{1}{2}, & x = \varphi \\ -\frac{1}{2}, & x = -\varphi. \end{cases}$

⑦ $f(x) = e^x$. $x \in [-\pi, \pi]$.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi} - e^{-\pi})$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx = \frac{1}{\pi}\cdot\frac{1}{1+n^2} e^x(n\sin nx + \cos nx)\Big|_{-\pi}^{\pi}$

$= \frac{(-1)^n}{\pi(1+n^2)}(e^{\pi} - e^{-\pi})$.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx = \frac{1}{\pi}\cdot\frac{1}{1+n^2} e^x(\sin nx - n\cos nx)\Big|_{-\pi}^{\pi}$.

$= \frac{(-1)^{n+1}\cdot n}{\pi(1+n^2)}(e^{\pi} - e^{-\pi})$.

令 $x = \pi$

$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2+1} = \pi\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} - \frac{1}{2}$.

$\Rightarrow e^x \sim \frac{e^{\pi}-e^{-\pi}}{2\pi} + \frac{e^{\pi}-e^{-\pi}}{\pi}\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+1}(\cos nx - \sin nx)$. $= \begin{cases} e^x, & x \in (-\pi, \pi) \\ \frac{e^{\pi}+e^{-\pi}}{2}, & x = \pm\pi. \end{cases}$

2. Parseval 等式.

如果 $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n \cos nx + b_n \sin nx)$ ，$x \in [-\pi, \pi]$.

则 $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{+\infty}(a_n^2 + b_n^2)$.

理解：Fourier 展开是函数 f 到序列 $\{a_0, a_1, b_1, a_2, b_2, \cdots\}$ 的映射. 映射前后，对象的"长度"一样. 即 Fourier 展开时是保距映射.

一般考察方式为给定 $f(x)$，求 Fourier 展开，再用 Parseval 等式求 $\sum_{n=1}^{+\infty}(a_n^2 + b_n^2) + \frac{a_0^2}{2}$ 的值.

① $\sum_{n=1}^{+\infty} \frac{\sin nx}{n}$ 记 $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\pi - x}{2}\right)^2 dx = \frac{\pi^2}{6}$

$\frac{\pi - x}{2} \sim$

$a_0 = -\frac{\pi^2}{3}$

② $\frac{x^2}{4} - \frac{\pi}{2}x \sim -\frac{\pi^2}{6} + \sum_{n=1}^{+\infty} \frac{\cos nx}{n^2}$.

记 $\sum_{n=1}^{+\infty} \frac{1}{n^4} + \frac{1}{2} \times \frac{\pi^4}{9} = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{x^2}{4} - \frac{\pi}{2}x\right)^2 dx = \frac{\pi^4}{90}$.

$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{\pi^4}{15} - \frac{\pi^4}{18} = \frac{\pi^4}{90}$.

③ $f(x) = \begin{cases} 1, & |x| \le \varphi \\ 0, & \varphi < |x| \le \pi \end{cases} \sim \frac{\varphi}{\pi} + \frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{\sin n\varphi}{n} \cdot \cos nx$. 求 $\sum_{n=1}^{+\infty} \frac{\cos n\varphi}{n^2}$

$\Rightarrow \frac{1}{\pi}\int_{-\varphi}^{\varphi} dx = \frac{1}{2}\cdot\left(\frac{2\varphi}{\pi}\right)^2 + \frac{4}{\pi^2}\sum_{n=1}^{+\infty} \frac{\sin^2 n\varphi}{n^2}$

$\frac{2\varphi}{\pi}$

$\Rightarrow \sum_{n=1}^{+\infty} \frac{\sin^2 n\varphi}{n^2} = \frac{\pi^2}{4}\cdot\left(\frac{2\varphi}{\pi} - \frac{2\varphi^2}{\pi^2}\right) = \frac{\pi\varphi}{2} - \frac{\varphi^2}{2}$.

又 $\sin^2 n\varphi = \frac{1 - \cos 2n\varphi}{2}$

故 $\frac{1}{2}\sum_{n=1}^{+\infty} \frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{+\infty} \frac{\cos 2n\varphi}{n^2} = \frac{\pi\varphi}{2} - \frac{\varphi^2}{2}$

故 $\sum_{n=1}^{+\infty} \frac{\cos 2n\varphi}{n^2} = \varphi^2 - \pi\varphi + \frac{\pi^2}{6}$

$\Rightarrow \sum_{n=1}^{+\infty} \frac{\cos n\varphi}{n^2} = (\frac{\varphi}{2})^2 - \pi\cdot\frac{\varphi}{2} + \frac{\pi^2}{6} = \frac{\varphi^2}{4} - \frac{\pi\varphi}{2} + \frac{\pi^2}{6}$.

④ 求 $\sum_{n=1}^{+\infty} \frac{\sin n\varphi}{n^3}$

「内积形式 Parseval 等式」.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$$

$$g(x) \sim \frac{c_0}{2} + \sum_{n=1}^{+\infty} (c_n \cos nx + d_n \sin nx)$$

则 $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx = \frac{a_0 c_0}{2} + \sum_{n=1}^{+\infty} (a_n c_n + b_n d_n)$

证明: $f(x) \pm g(x) \sim \frac{a_0 \pm c_0}{2} + \sum_{n=1}^{+\infty} ((a_n \pm c_n)\cos nx + (b_n \pm d_n)\sin nx)$.

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} (f+g)^2\,dx = \frac{(a_0+c_0)^2}{2} + \sum_{n=1}^{+\infty}\left((a_n+c_n)^2 + (b_n+d_n)^2\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f-g)^2\,dx = \frac{(a_0-c_0)^2}{2} + \sum_{n=1}^{+\infty}\left((a_n-c_n)^2 + (b_n-d_n)^2\right).$$

作差即得. □.

用 $\frac{x^2}{4} - \frac{\pi}{2}x \sim -\frac{\pi^2}{6} + \sum_{n=1}^{+\infty} \frac{\cos nx}{n^2}$

$$\begin{cases} 1, & |x| \le \varphi \\ 0, & \varphi < |x| \le \pi \end{cases} \sim \frac{\varphi}{\pi} + \frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{\sin n\varphi}{n}\cdot\cos nx$$

有 $\frac{2}{\pi}\int_0^{\varphi} (\frac{x^2}{4} - \frac{\pi}{2}x)\,dx = \frac{1}{2}\cdot(-\frac{\pi^2}{3})\cdot\frac{2\varphi}{\pi} + \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{\sin n\varphi}{n^3}$

即 $\frac{\varphi^3}{6\pi} - \frac{\varphi^2}{2} = -\frac{\pi\varphi}{3} + \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{\sin n\varphi}{n^3}$

$$\sum_{n=1}^{+\infty} \frac{\sin n\varphi}{n^3} = \frac{\varphi^3}{12} - \frac{\varphi^2\pi}{4} + \frac{\varphi\pi^2}{6}. \quad \varphi \in (0,\pi)$$

(对 $\varphi \in [0,\pi]$ 实际也对).

3. Fourier 级数的一致收敛性.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx).$$

「定理」. $f(x)$ 以 2π 为周期, $f'(x) \in R[-\pi, \pi]$.

则 Fourier 级数在 $\mathbb{R}$ 上 $\rightrightarrows f(x)$.

证明: 设 $f'(x) \sim \frac{a_0'}{2} + \sum_{n=1}^{+\infty} (a_n' \cos nx + b_n' \sin nx)$

$$a_0' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,dx = \frac{1}{\pi}(f(\pi) - f(-\pi)) = 0.$$

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\,df(x)$$

$$= \frac{1}{\pi}\underbrace{f(x)\cos nx\Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,d\cos nx$$

$$= \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = n b_n.$$

$$b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\,df(x)$$

$$= \frac{1}{\pi}\underbrace{f(x)\sin nx\Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,d\sin nx$$

$$= -\frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = -n a_n.$$

故 $f'(x) \sim \sum_{n=1}^{+\infty} (n b_n \cdot \cos nx - n a_n \cdot \sin nx)$.

Rmk. 这正是 $\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$ 逐项微分!

由 Parseval 等式.

$$\sum_{n=1}^{+\infty} (|a_n| + |b_n|) = \sum_{n=1}^{+\infty} \frac{1}{n}(|a_n'| + |b_n'|).$$

$$\le \sum_{n=1}^{+\infty} \frac{1}{n^2} \cdot \sum_{n=1}^{+\infty} (|a_n'|^2 + |b_n'|^2)$$

$$= \frac{\pi^2}{6} \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} |f'(x)|^2\,dx < +\infty.$$

由 M-判别法. 知 $\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) \rightrightarrows f(x)$. □.

4. Fourier 级数的唯一性.

$f(x)$ 和 $g(x)$ 对应相同 Fourier 级数 $\Leftrightarrow \int_{-\pi}^{\pi} |f(x)-g(x)|^2 dx = 0$.

证明: 设 $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$

$$g(x) \sim \frac{c_0}{2} + \sum_{n=1}^{+\infty} (c_n \cos nx + d_n \sin nx)$$

$$f(x)-g(x) \sim \frac{a_0-c_0}{2} + \sum_{n=1}^{+\infty} \left((a_n-c_n)\cos nx + (b_n-d_n)\sin nx \right)$$

则 $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|^2 dx = \frac{1}{2}(a_0-c_0)^2 + \sum_{n=1}^{+\infty} \left((a_n-c_n)^2 + (b_n-d_n)^2 \right)$.

$LHS=0 \Leftrightarrow RHS=0 \Leftrightarrow a_n=c_n,\ b_n=d_n,\ \forall n$. □.

Rmk. Taylor 级数不具有这种唯一性.

例如 $f(x) \sim \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$.

由于 $\begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x=0 \end{cases}$ 在 0 处任意阶导为 0.

故 $f(x) + e^{-\frac{1}{x^2}} \sim \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$.

但 $|f(x)+e^{-\frac{1}{x^2}} - f(x)| = |e^{-\frac{1}{x^2}}| \neq 0$.

5. Poincaré 不等式.

$f \in C^1[0,2\pi]$. $f(0)=f(2\pi)$. $\int_0^{2\pi} f(x)\,dx = 0$. 则 $\int_0^{2\pi} (f'(x))^2 dx \geq \int_0^{2\pi} f^2(x)\,dx$

证明: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$ $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$.

$$f'(x) \sim \sum_{n=1}^{+\infty} (n b_n \cos nx - n a_n \sin nx)$$

$$LHS = \pi \cdot \sum_{n=1}^{+\infty} (n^2 b_n^2 + n^2 a_n^2) \geq \pi \cdot \sum_{n=1}^{+\infty} (a_n^2 + b_n^2) = RHS. \quad \square.$$

Rmk. 由证明还知道取等号 $\Leftrightarrow f(x) = A\sin x + B\cos x$.

一个有趣的应用是等周不等式. L 是简单闭曲线 C 的长度, A 为内部面积

则 $L^2 \geq 4\pi A$. 取等 $\Leftrightarrow C$ 是圆. t 为弧长, $L=2\pi$ 那么, $(x'(t))^2+(y'(t))^2 = \frac{dx^2+dy^2}{dt^2} = 1$.

证明: 参数化 C $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ $t \in [0,2\pi]$. $x(0)=x(2\pi)$, $y(0)=y(2\pi)$

并平移使 $\int_0^{2\pi} x(t)\,dt = \int_0^{2\pi} y(t)\,dt = 0$.

那么, $L = \int_0^{2\pi} \sqrt{(x'(t))^2+(y'(t))^2}\,dt$ $\quad \Rightarrow A \leq \pi = \frac{L^2}{4\pi}$ □.

$$4A^2 = \left(\int_0^{2\pi} (xy' + x'y)\,dt\right)^2 \leq \int_0^{2\pi} x^2+y^2\,dt \int_0^{2\pi} x'^2+y'^2\,dt \leq \left(\int_0^{2\pi} x'^2+y'^2\,dt\right)^2 = 4\pi^2$$

6. 一道往年问题.

① 求 $\cos\alpha x$, $\alpha \in (0,1)$ Fourier 展开. $x \in [-\pi, \pi]$.

$b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x\, dx = \frac{1}{\alpha\pi}\sin\alpha x\Big|_{-\pi}^{\pi} = \frac{2\sin\alpha\pi}{\alpha\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} \cos\alpha x\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(\alpha+n)x + \cos(\alpha-n)x\, dx$$

$$= \frac{1}{\pi}\left(\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}\right).$$

故 $\cos\alpha x \sim \frac{\sin\alpha\pi}{\alpha\pi} + \frac{1}{\pi}\sum_{n=1}^{+\infty}\left(\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}\right)\cos nx$

$= \cos\alpha x, \quad x \in [-\pi, \pi]$ (因为 $\cos\alpha\pi = \cos(\alpha\cdot(-\pi))$).

② 求证: $\frac{\pi}{\sin\alpha\pi} = \frac{1}{\alpha} + \sum_{n=1}^{+\infty}(-1)^n\frac{2\alpha}{\alpha^2-n^2}$.

对上式中取 $x = 0$. ~~[illegible]~~ $1 = \frac{\sin\alpha\pi}{\alpha\pi} + \frac{1}{\pi}\sum_{n=1}^{+\infty}(-1)^n\left(\frac{\sin\alpha\pi}{\alpha+n} + \frac{\sin\alpha\pi}{\alpha-n}\right)$

变形: $\frac{\pi}{\sin\alpha\pi} = \frac{1}{\alpha} + \sum_{n=1}^{+\infty}(-1)^n\frac{2\alpha}{\alpha^2-n^2}$ □.

③ 求证: $\frac{\pi}{\sin x} = \frac{\pi}{x} + \sum_{n=1}^{+\infty}(-1)^n\frac{2\pi x}{x^2-n^2\pi^2}$.

证明: 令 $\alpha = \frac{x}{\pi}$ 即得 □.

④ 求 $\int_0^{+\infty}\frac{\sin x}{x}dx$.

解: $\frac{\text{取}1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{+\infty}(-1)^n\left(\frac{1}{x-n\pi} + \frac{1}{x+n\pi}\right)$.

$$\frac{\sin x}{x} = 1 - \sum_{n=1}^{+\infty}(-1)^n\left(\frac{\sin x}{x-n\pi} + \frac{\sin x}{x+n\pi}\right)$$

$$= 1 - \sum_{n=1}^{+\infty}\left(\frac{\sin(x-n\pi)}{x-n\pi} + \frac{\sin(x+n\pi)}{x+n\pi}\right).$$

即 $1 = \sum_{n=-\infty}^{+\infty}\frac{\sin(x+n\pi)}{x+n\pi}$ (Dirichlet 判别和一致收敛)

故 $\pi = \int_0^{\pi}dx = \sum_{n=-\infty}^{+\infty}\int_0^{\pi}\frac{\sin(x+n\pi)}{x+n\pi}dx = \sum_{n=-\infty}^{+\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin x}{x}dx = \int_{-\infty}^{+\infty}\frac{\sin x}{x}dx$

由偶性, $\int_0^{+\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$.

7. 逼近定理介绍.

Parseval 等式说明 $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n \cos nx + b_n \sin nx) \triangleq S(x)$

$S(x)$ 在 L^2 意义下收敛于 $f(x)$. 即 $\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x) - \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) - \frac{a_0}{2}]^2 dx \to 0$.

那么是否可以用三角多项式做到一致逼近 $f(x)$ 呢?

「第二逼近定理」. $f(x) \in C[0, 2\pi]$. 以 2π 为周期. 则存在三角多项式序列 $T_n(x)$ s.t. $T_n(x) \rightrightarrows f(x)$. $x \in [0, 2\pi]$.

但 $T_n(x)$ 不是 $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$.

特别地, 可取 $T_n(x) = \frac{1}{n}(S_1(x) + S_2(x) + \cdots + S_n(x))$ 能够满足要求.

「第一逼近定理」. $f(x) \in C[a, b]$. 则存在多项式序列 $P_n(x)$ s.t. $P_n(x) \rightrightarrows f(x)$.

同样地, $P_n(x)$ 不是 Taylor 多项式.

一般可以对 $f(x) \in C[0, 1]$ 取 $P_n(x) = \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k}$ 可满足要求. 称为 Bernstein 多项式.

(*) 高数中, 我们遇到三种逼近, 它们意义不同.

① $f(x) \in C^n(x_0)$. 则在 x_0 处对 f 做 Taylor 展开, $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.

有 $f(x) - T_n(x) = o((x - x_0)^n)$. $(x \to x_0)$. 这是局部最佳逼近.

② $f(x) \in C[a, b]$. 找一列 $f_n(x)$ s.t. $f_n \rightrightarrows f$. $x \in [a, b]$.

这是一致逼近, 是在距离 ~~$f - g$~~ $\max_{a \le x \le b} |f(x) - g(x)|$ 的意义下逼近.

③ $f(x) \in C[0, 2\pi]$. Fourier 展开 $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$

有 $\lim_{n \to \infty} \frac{1}{\pi}\int_0^{2\pi}(f(x) - S_n(x))^2 dx = 0$. 这是 L^2 逼近, 是在距离 $\int_0^{2\pi}|f(x) - g(x)|^2 dx$ 的意义下逼近.