2026.3.24. 曲线积分与 Green 公式.

1. 微分形式简介.

假设底空间 $\mathbb{R}^n$. 定义 k-形式 (k-form)

$$\omega = \sum_{\alpha} a_\alpha \, dx_{\alpha_1} \wedge dx_{\alpha_2} \cdots \wedge dx_{\alpha_k}$$

α 为取值于 $\{1, 2, \cdots, n\}^k$ 的向量.

Eg. $\mathbb{R}^3$ 中的 0-形式 $f(x), g(x), \cdots$ 即函数.

1-形式 $f(x)\,dx_1, \cdots, f(x)\,dx_2, \cdots$

2-形式 $f(x,y,z)\,dx\wedge dy$, $f(x,y,z)\,dz\wedge dx$

3-形式 $f(x,y,z)\,dx\wedge dy\wedge dz$.

[定义.] (wedge)

对 2 个 k-形式 ω, η, 定义其运算 $\omega\wedge\eta$. 满足.

① 线性性

$$(a\omega_1 + b\omega_2)\wedge\eta = a\omega_1\wedge\eta + b\omega_2\wedge\eta.$$

② 反交换性

& $\omega\wedge\eta = -\eta\wedge\omega$.

Eg. $dx\wedge dy = -dy\wedge dx$, $dx\wedge dx = -dx\wedge dx \Rightarrow dx\wedge dx = 0$.

[定义.] (微分算子)

d 把一个 k 形式映射成一个 k+1 形式. 满足

① 线性性

$$d(a\omega+b\eta) = a\,d\omega + b\,d\eta.$$

② 如果 ω 形式如 $f(x_1,\cdots,x_n)\,dx_{i_1}\wedge dx_{i_2}\cdots\wedge dx_{i_k}$

则 $d\omega \triangleq \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,dx_i\right)\wedge\left(dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_k}\right)$

Rmk. 我们只会用到这个. 一般的 d 运算还有其它规则.

Eg. 在 $\mathbb{R}^2$ 中. $\omega = P\,dx + Q\,dy$

$$d\omega = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)\wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)\wedge dy$$

$$= \frac{\partial P}{\partial x}\underbrace{dx\wedge dx}_{=0} + \frac{\partial P}{\partial y}\underbrace{dy\wedge dx}_{=-dx\wedge dy} + \frac{\partial Q}{\partial x}dx\wedge dy + \frac{\partial Q}{\partial y}\underbrace{dy\wedge dy}_{=0}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy.$$

在 $\mathbb{R}^3$ 中. ~~$\omega = P\,dx\wedge dy + Q\,dy\wedge dx + R\,d$~~

$$\omega = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy.$$

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\wedge dy\wedge dz.$$

在 $\mathbb{R}^3$ 中. $\omega = P\,dx + Q\,dy + R\,dz.$

$$d\omega = \frac{\partial P}{\partial x}\underbrace{dx\wedge dx}_{0} + \frac{\partial P}{\partial y}dy\wedge dx + \frac{\partial P}{\partial z}dz\wedge dx$$

$$+ \frac{\partial Q}{\partial x}dx\wedge dy + \frac{\partial Q}{\partial y}\underbrace{dy\wedge dy}_{0} + \frac{\partial Q}{\partial z}dz\wedge dy$$

$$+ \frac{\partial R}{\partial x}dx\wedge dz + \frac{\partial R}{\partial y}dy\wedge dz + \frac{\partial R}{\partial z}\underbrace{dz\wedge dz}_{0}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\wedge dz$$

$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\wedge dx$$

[广义 Stokes 定理].

D 是区域, ∂D 为其边界. ω 是一个微分形式. 则

$$\int_D d\omega = \int_{\partial D}\omega.$$

例子. $\mathbb{R}^2$: $\omega = P\,dx + Q\,dy \Rightarrow \oint_\Gamma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$ (Green 公式.)

$\mathbb{R}^3$: $\omega = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy \Rightarrow \oint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \int_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\,dy\,dz$

(Gauss 公式)

$\mathbb{R}^3$: $\omega = P\,dx + Q\,dy + R\,dz \Rightarrow \oint_\Gamma P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\,dx$

(Stokes 公式).

$\mathbb{R}$: $\omega = f'(x)\,dx \Rightarrow \int_a^b f'(x)\,dx = \int_{\{a,b\}} f(x) = f(b) - f(a)$

(Newton-Leibniz 公式).

2. Green 公式及应用.

叙述. $D\subset\mathbb{R}^2$ 有界闭区域，∂D 是分段 C^1 曲线. 若 P,Q 在 D 上 C^1，则 $\int_{\partial D} P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$

↓

如果 P,Q 在 D 上某点奇异，则 Green 公式不对！

如 $P=\frac{x}{x^2+y^2}$ 则 D 不能包含 $(0,0)$！

① 型曲线积分简化计算.

(i) $I=\int_C(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy$

$C: 2x=\pi y^2$. $(0,0)\to(\frac{\pi}{2},1)$.

计算 $\frac{\partial}{\partial y}(2xy^3-y^2\cos x)=6xy^2-2y\cos x$

$\frac{\partial}{\partial x}(1-2y\sin x+3x^2y^2)=-2y\cos x+6xy^2$

故可将 C 换为另一条路径 ☆. 路径无关性.

取 Γ: $(0,0)\xrightarrow{dy=0}(\frac{\pi}{2},0)\xrightarrow{dx=0}(\frac{\pi}{2},1)$.

$$I=\int_0^{\frac{\pi}{2}}(0-0)\,dx+\int_0^1\left(1-2y\sin\frac{\pi}{2}+3\times(\frac{\pi}{2})^2y^2\right)dy$$

$$=\int_0^1 1-2y+\frac{3}{4}\pi^2y^2\,dy=1-1+\frac{\pi^2}{4}=\frac{\pi^2}{4}.$$

(ii) $I=\int_\Gamma 2xy\,dx+(x^2-y^2)\,dy$. Γ: ~~$(-1,0)\to(1,0)$~~ $(1,0)\to(0,1)$. 圆弧.

(a.) 极坐标法. $\begin{cases}x=\cos\theta\\ y=\sin\theta.\end{cases}$ $\theta\in[0,\frac{\pi}{2}]$.

$$I=\int_0^{\frac{\pi}{2}}2\sin\theta\cos\theta\cdot(-\sin\theta)\,d\theta+(\cos^2\theta-\sin^2\theta)\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}(\cos^2\theta-3\sin^2\theta)\,d\sin\theta\overset{t=\sin\theta}{=\!=}\int_0^1(1-4t^2)\,dt=1-\frac{4}{3}=-\frac{1}{3}.$$

(6.) $\frac{\partial}{\partial y}(2xy) = 2x = \frac{\partial}{\partial x}(x^2-y^2)$

故取另一条路径 C: $(1,0) \to (0,0) \to (0,1)$

$$I = \int_1^0 0\,dx + \int_0^1 -y^2\,dy = -\frac{1}{3}.$$

② 有"洞"的情形.

(i) $a,b,c \in \mathbb{R}$, $ac-b^2>0$. $I = \int_C \frac{x\,dy - y\,dx}{ax^2+2bxy+cy^2}$ C为绕$(0,0)$的任一简单闭曲线.

解: $\frac{\partial}{\partial x}\left(\frac{x}{ax^2+2bxy+cy^2}\right) = \frac{ax^2+2bxy+cy^2-2ax^2-2bxy}{(ax^2+2bxy+cy^2)^2} = \frac{cy^2-ax^2}{(ax^2+2bxy+cy^2)^2}$

$\frac{\partial}{\partial y}\left(\frac{-y}{ax^2+2bxy+cy^2}\right) = \frac{cy^2-ax^2}{(ax^2+2bxy+cy^2)^2}$

Rmk. 这并不能说明 $I=0$. 因为 $(0,0)$在C内部, 而在$(0,0)$, 函数没有定义!

故可特殊地取Γ为 $ax^2+2bxy+cy^2=1$. 椭圆圈.

由 Green公式. $\int_C \bigcirc = \int_\Gamma \bigcirc$

在这上面的



$\Rightarrow I = \int_\Gamma \frac{x\,dy-y\,dx}{ax^2+2bxy+cy^2} = \int_\Gamma x\,dy - y\,dx.$ 不同消失了!

$\frac{\partial x}{\partial x} = \frac{\partial x}{\partial y}$

Green公式 $\iint_{\Gamma\text{内部}} 2\,dx\,dy$

(= 2倍椭圆面积.)

换元 $1 = ax^2+2bxy+cy^2 = \left(\sqrt{a}\left(x+\frac{b}{a}y\right)\right)^2 + \left(\sqrt{c-\frac{b^2}{a}}\,y\right)^2$

$$\begin{cases} u = \sqrt{a}\left(x+\frac{b}{a}y\right) \\ v = \sqrt{c-\frac{b^2}{a}}\,y. \end{cases} \quad \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \sqrt{a} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{c-\frac{b^2}{a}} \end{vmatrix} = \sqrt{ac-b^2}.$$

$$\Rightarrow 2\iint_{\Gamma\text{内部}} dx\,dy = 2\int_{u^2+v^2\le 1} \frac{du\,dv}{\sqrt{ac-b^2}} = \frac{2\pi}{\sqrt{ac-b^2}}.$$

$$\Rightarrow I = \frac{2\pi}{\sqrt{ac-b^2}}$$

(ii) $I = \oint_{x^2+y^2=1} \frac{e^y}{x^2+y^2}\left((x\sin x + y\cos x)\,dx + (y\sin x - x\cos x)\,dy\right)$

$$\frac{\partial}{\partial y}\left(\frac{e^y}{x^2+y^2}(x\sin x + y\cos x)\right) = \frac{1}{(x^2+y^2)^2}\left(e^y\cdot\underset{(y+1)\cdot\cos x}{\cancel{\cos x}}\cdot(x^2+y^2) - 2y e^y(x\sin x + y\cos x)\right)$$

$$\frac{\partial}{\partial x}\left(\frac{e^y}{x^2+y^2}(y\sin x - x\cos x)\right) = \frac{1}{(x^2+y^2)^2}\Big(\cancel{e^{(y+1)\cos x}} \;\|\; (e^y y\cos x - e^y\cos x + e^y x\sin x)(x^2+y^2) - 2x e^y(y\sin x - x\cos x)\Big).$$

故 $\forall \varepsilon > 0$. $\oint_{x^2+y^2=1} = \oint_{x^2+y^2=\varepsilon^2}$.

$$I = I_\varepsilon = \oint_{x^2+y^2=\varepsilon^2} \frac{e^y}{\varepsilon^2}\Big((x\sin x + y\cos x)\,dx + (y\sin x - x\cos x)\,dy\Big).$$

因三有再次出0不同.

$$\frac{\partial}{\partial y}\left(e^y(x\sin x + y\cos x)\right) = e^y(x\sin x + y\cos x + \cos x).$$

$$\frac{\partial}{\partial x}\left(e^y(y\sin x - x\cos x)\right) = e^y(y\cos x - \cos x + x\sin x)$$

$$\Rightarrow I_\varepsilon = \frac{1}{\varepsilon^2}\iint_{x^2+y^2<\varepsilon^2} -2e^y\cos x\,dx\,dy.$$

$$= \frac{1}{\varepsilon^2}\cdot\pi\varepsilon^2\cdot\left(-2e^{\xi_\varepsilon}\cos\eta_\varepsilon\right) \quad \text{其中 } (\xi_\varepsilon,\eta_\varepsilon)\in\{x^2+y^2<\varepsilon^2\} \text{ 依赖于 } \varepsilon.$$

$$= -2\pi\cdot\left(e^{\xi_\varepsilon}\cos\eta_\varepsilon\right) \xrightarrow{\varepsilon\to 0} -2\pi$$

但 I_ε 与 ε 无关. 故 $I_\varepsilon = -2\pi \quad \forall \varepsilon > 0$.

$$\Rightarrow I = -2\pi.$$

Rmk. 当积为二重积分或曲线足够简单，却还是显然积不出来时，可能需要这种技术. 让曲线缩小并计算极限.

③ 面积.

设 Γ 为 D 边界. 用 Green 公式. $\iint_D dxdy = \frac{1}{2}\oint_\Gamma x\,dy - y\,dx$

$S(D) = \oint_\Gamma x\,dy = \oint_\Gamma -y\,dx$.

Eg. ~~$x^3+y^3=3axy$~~

$\begin{cases} x = \cos^3\theta \\ y = \sin^3\theta. \end{cases}$ $\theta \in [0, 2\pi]$ 围成面积.

$$S = \int_0^{2\pi} \cos^3\theta\, d\sin^3\theta = \int_0^{2\pi} 3\cos^3\theta\cdot\sin^2\theta\cos\theta\, d\theta$$
$$= 3\int_0^{2\pi} \cos^4\theta(1-\cos^2\theta)\, d\theta = 3\int_0^{2\pi}(\cos^4\theta - \cos^6\theta)\, d\theta$$
$$= 3\left(\frac{3}{4}\pi - \frac{5}{8}\pi\right) = \frac{3}{8}\pi.$$

④ 余弦形式.

$\oint_\Gamma (P\cos(\vec{n},x) + Q\cos(\vec{n},y))\,ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) d\sigma$. $\vec{n}$ 为 Γ 外法向



证明: ~~$\cos(\vec{n},x)\,ds$~~ $\vec{L} =$

在 (x,y) 处切向是 ~~(dx, dy)~~. $(\dot{x}(t), \dot{y}(t))$

故法向 $\vec{n} = \left(\frac{+\dot{y}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}}, \frac{-\dot{x}(t)}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}}\right)$.

关系应为 $\vec{L}$ 顺转 90° 变成 $\vec{n}$.

故 $LHS = \oint_\Gamma P\,dy - Q\,dx \overset{\text{Green}}{=\!=} \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) d\sigma$. □.

⑤ 散度定理

将以上形式改写.

$LHS = \oint_\Gamma (P,Q)\cdot\vec{n}\,ds$, $RHS = \iint_D \nabla\cdot(P,Q)\,dxdy$.

(P,Q) 可以看作一个二元函数 $v: \mathbb{R}^2 \to \mathbb{R}^2$.

即有 $\oint_\Gamma \vec{v}\cdot\vec{n}\,ds = \iint_D \nabla\cdot\vec{v}\,dxdy$.

将 $\vec{v}$ 理解为平面上一个流速场. 上述公式是说, 水流在 Γ 上的通量是 $\vec{v}$ 的散度的积分.

⑥ 二维分部积分.

习题 8.3.12 . $\vec{n}$ 为 Γ 外法向.

(1) $\iint_D v\triangle u\,d\sigma = \oint_\Gamma v\frac{\partial u}{\partial \vec{n}}ds - \iint_D \nabla u\cdot\nabla v\,d\sigma$. (第一Green恒等式)

证明: $\oint_\Gamma v\frac{\partial u}{\partial \vec{n}}ds = \oint_\Gamma (v\nabla u)\cdot\vec{n}\,ds$

$\overset{\text{Green}}{=\!=} \iint_D \nabla\cdot(v\nabla u)\,d\sigma$.

$= \iint_D (\nabla v\cdot\nabla u + v\triangle u)\,d\sigma$. □.

(2) $\iint_D (u\triangle v - v\triangle u)\,d\sigma = \oint_L (u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}})ds$ (第二Green恒等式)

证明: 由(1). $\iint_D v\triangle u\,d\sigma = \oint_\Gamma v\frac{\partial u}{\partial \vec{n}}ds - \iint_D \nabla u\cdot\nabla v\,d\sigma$

$\iint_D u\triangle v\,d\sigma = \oint_\Gamma u\frac{\partial v}{\partial \vec{n}}ds - \iint_D \nabla u\cdot\nabla v\,d\sigma$

作差即证 □.

〈高维分部积分公式〉

Ω 是区域, $\partial\Omega$ 是其边界, $\vec{n}$ 为 $\partial\Omega$ 外单位法向.

$u:\Omega\to\mathbb{R}$, $\vec{v}:\Omega\to\mathbb{R}^n$. 则 $\int_\Omega u(\nabla\cdot\vec{v})dV = \int_{\partial\Omega} u(\vec{v}\cdot\vec{n})dS - \int_\Omega \nabla u\cdot\vec{v}\,dV$

$n=2$ 情况下, $\int_{\partial\Omega} u(\vec{v}\cdot\vec{n})dS = \int_{\partial\Omega} (uv_1\cos(\vec{n},x) + uv_2\cos(\vec{n},y))ds$

$= \iint_\Omega \frac{\partial}{\partial x}(uv_1) + \frac{\partial}{\partial y}(uv_2)\,dxdy$

$= \iint_\Omega (\partial_x u\cdot v_1 + \partial_y u\cdot v_2)\,dxdy + \iint_\Omega u(\partial_x v_1 + \partial_y v_2)dxdy$

$= \iint_\Omega \nabla u\cdot\vec{v}\,dV + \iint_\Omega u(\nabla\cdot\vec{v})\,dV$.

对于一般的 n 都成立. Green 公式也称为分部积分公式.

① 向量场的旋转.

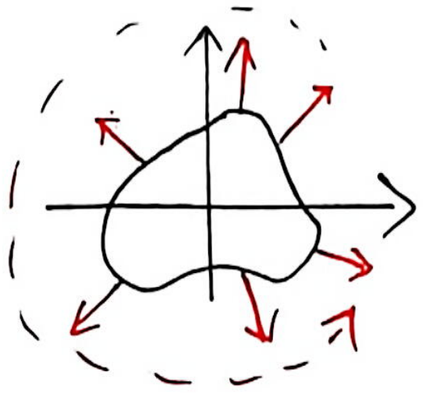
"为什么 $\oint_\Gamma \frac{x\mathrm{d}y-y\mathrm{d}x}{x^2+y^2} = 2\pi$? 其中 $(0,0)$ 在 Γ 内部".

考虑 $(P(x,y), Q(x,y)): \mathbb{R}^2 \to \mathbb{R}^2$ 是 C^1 的. 那么 (P,Q) 是 $\mathbb{R}^2$ 上一个向量场. 沿着一条曲线 Γ 绕一圈, 这个向量场旋转了多少度?

$$2\pi\alpha = \oint_\Gamma \mathrm{d}\arctan\frac{Q}{P} = \oint_\Gamma \frac{P\mathrm{d}Q-Q\mathrm{d}P}{P^2+Q^2}.$$ 这称为"旋转数".

由于转一圈会回到起始, 故必有 $\alpha \in \mathbb{Z}$.

Eg. $(P,Q) = (x,y)$.



明显旋转一圈. $\alpha = 1$.

Eg. $(P,Q) = (x^2-y^2, 2xy)$. $\alpha = 2$.

Eg. (切线回转定理). $\Gamma\begin{cases} x(t) \\ y(t) \end{cases}$

$$\oint_\Gamma \frac{\dot{x}(t)\mathrm{d}\dot{y}(t)-\dot{y}(t)\mathrm{d}\dot{x}(t)}{\dot{x}^2(t)+\dot{y}^2(t)} = \text{整数} \times 2\pi$$

3. 往年题.

① 2023.4.

$$I(R) = \oint_{x^2+y^2=R^2} \frac{x\mathrm{d}y-y\mathrm{d}x}{(x^2+xy+y^2)^2}$$ 求证: $\lim\limits_{R\to+\infty} I(R) = 0$

证明用: $R^2 \le x^2+xy+y^2 \le \frac{3}{2}R^2$

$$\Rightarrow \frac{4}{9R^4}\oint_{x^2+y^2=R^2} x\mathrm{d}y-y\mathrm{d}x \le I(R) \le \frac{1}{R^4}\oint_{x^2+y^2=R^2} x\mathrm{d}y-y\mathrm{d}x$$

$$\xRightarrow{\text{面积}} \frac{4}{9R^4}\cdot 2\pi R^2 \le I(R) \le \frac{1}{R^4}\cdot 2\pi R^2$$

$$\Rightarrow \frac{8}{9}\cdot\frac{\pi}{R^2} \le I(R) \le \frac{2\pi}{R^2}. \Rightarrow \lim_{R\to+\infty} I(R) = 0. \quad \square.$$

2023.9. 求 $I=\oint_\Gamma \left(\frac{y^2+y+4x^2}{4x^2+y^2}+\sin x^2\right)dx+\left(\frac{4x^2-x+y^2}{4x^2+y^2}+\sin y^2\right)dy$

Γ: $x^2+y^2=4\ (y\geq 0)$ 和 $\frac{x^2}{4}+\frac{y^2}{16}=1\ (y\leq 0)$

P Q.

解: 易验证 $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{4x^2-y^2}{(4x^2+y^2)^2}$ 奇点仅 $(0,0)$.

故选取圆道 $4x^2+y^2=4$. 即 $x^2+\frac{y^2}{4}=1$ 记为 C.

由 Green 公式. $I=\oint_C P\,dx+Q\,dy$

$=\iint_{x^2+\frac{y^2}{4}\leq 1}\frac{4x^2-y^2}{(4x^2+y^2)^2}\,dx\,dy$ 洞在洞.

$=\oint_{x^2+\frac{y^2}{4}=1}\left(\frac{1}{4}(y^2+y+4x^2)+\sin x^2\right)dx+\left(\frac{1}{4}(4x^2-x+y^2)+\sin y^2\right)dy$

$\frac{\partial}{\partial y}\left(\frac{1}{4}(y^2+y+4x^2)+\sin x^2\right)=\frac{1}{2}y+\frac{1}{4}$

$\frac{\partial}{\partial x}\left(\frac{1}{4}(4x^2-x+y^2)+\sin y^2\right)=\frac{1}{2}x-\frac{1}{4}$.

$\xlongequal{\text{Green}}\iint_{x^2+\frac{y^2}{4}\leq 1}\left(\left(\frac{1}{2}x-\frac{1}{4}\right)-\left(\frac{1}{2}y+\frac{1}{4}\right)\right)dx\,dy$

$=\frac{1}{2}\iint_{x^2+\frac{y^2}{4}\leq 1}(x-y-1)\,dx\,dy$.

极坐换元 $\begin{cases}x=r\cos\theta\\ y=2r\sin\theta\end{cases}$ 由于对称性 $\iint x$, $\iint y=0$.

$=-\frac{1}{2}\iint_{x^2+\frac{y^2}{4}\leq 1}dx\,dy=-\frac{1}{2}\times 2\pi=-\pi$.