2026.4.7. ODE I.

1. 初等积分求解 ODE.

① $y' = \frac{x+y}{x-y}$.

齐 这是关于 x, y 的齐次函数.

令 $z = \frac{y}{x}$. $y = xz$. $y' = z + xz'$.

原方程化为 $x\frac{dz}{dx} = \frac{z^2+1}{1-z}$

$\Rightarrow \frac{1-z}{z^2+1}dz = \frac{dx}{x}$ $\Rightarrow$ $\arctan z = C + \ln|x|\cdot\sqrt{1+z^2}$

$= C + \ln\sqrt{x^2+y^2}$

$\Rightarrow \sqrt{x^2+y^2} = Ce^{\arctan\frac{y}{x}}$ $(C>0)$.

(法2.) 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. 原式变成 $dr + rd\theta = 0$.

$\Rightarrow r = Ce^{\theta}$ $(C>0)$

即 $\sqrt{x^2+y^2} = Ce^{\arctan\frac{y}{x}}$ $(C>0)$,

该轨迹称为螺线.

② $2x^3y' = (2x^2-y^2)y$. $(x>0)$.

$y' = \frac{(2x^2-y^2)y}{2x^3}$. 这是齐次函数.

令 $u = \frac{y}{x}$. $y' = u + xu'$

$\Rightarrow$ ~~$2u + 2xu$~~ $2x^3u + 2x^4u' = (2-u^2)ux^3$

$\Rightarrow 2xu' = -u^3 \Rightarrow \frac{dx}{x} = -\frac{2du}{u^3}$

$\Rightarrow C + \ln x = \frac{1}{u^2} \Rightarrow x = Ce^{x^2/y^2}$. $(C>0)$

$(x>0)$

⑨ $y' = \dfrac{-x+2y+5}{2x-y-4}$.

这是形如 $y' = f\left(\dfrac{ax+by+c}{dx+ey+f}\right)$ 且 $\begin{vmatrix} a & b \\ d & e \end{vmatrix} \neq 0$.

~~$\dfrac{dy}{dx} = x$~~ 因此 b，方程 $\begin{cases} -x+2y+5=0 \\ 2x-y-4=0 \end{cases}$ 有解 $\begin{cases} x=1 \\ y=-2 \end{cases}$ 平移后齐次.

即 $\begin{cases} -(x-1)+2(y+2) = -x+2y+5 \\ 2(x-1)-(y+2) = 2x-y-4. \end{cases}$

故换元 $\begin{cases} u = x-1 \\ v = y+2. \end{cases}$ $y' = \dfrac{dy}{dx} = \dfrac{dv}{du}$

$\Rightarrow \dfrac{dv}{du} = \dfrac{2v-u}{2u-v}$ 是齐次函数.

令 $t = \dfrac{v}{u}$. $v = tu$. $\dfrac{dv}{du} = t + u\dfrac{dt}{du}$

$\Rightarrow t + u\dfrac{dt}{du} = \dfrac{2t-1}{2-t} \Rightarrow u\dfrac{dt}{du} = \dfrac{2t-1-2t+t^2}{2-t} = \dfrac{t^2-1}{2-t}$

$\Rightarrow \dfrac{(2-t)\,dt}{t^2-1} = \dfrac{du}{u}$ 两侧积分. $\int \dfrac{du}{u} = \ln|u| + C$.

$\int \dfrac{(2-t)\,dt}{t^2-1} = \int \dfrac{2dt}{t^2-1} - \int \dfrac{t\,dt}{t^2-1}$

$= \int \left(\dfrac{1}{t-1} - \dfrac{1}{t+1}\right) dt - \dfrac{1}{2}\int \dfrac{dt^2}{t^2-1} = \ln\left|\dfrac{t-1}{t+1}\right| - \dfrac{1}{2}\ln|t^2-1|.$

$= \ln\sqrt{\left|\dfrac{t-1}{(t+1)^3}\right|}$.

因此 $\ln\sqrt{\left|\dfrac{t-1}{(t+1)^3}\right|} = \ln|u| + C$. 两侧 $\times 2$ 再取指数

有绝对值打开，有可能负.

$\Rightarrow \dfrac{t-1}{(t+1)^3} = Cu^2$. $(C \neq 0)$. Recall. $u = x-1$

$t = \dfrac{y+2}{x-1}$

$t-1 = \dfrac{-x+y+3}{x-1}$

$t+1 = \dfrac{x+y+1}{x-1}$

$\Rightarrow (x-1)^2 \cdot \dfrac{-x+y+3}{(x+y+1)^3} = C(x-1)^2$

$\Rightarrow -x+y+3 = C\cdot(x+y+1)^3$. $(C \neq 0)$

一般会这样.

另检查 $C=0$. 和 $x+y+1=0$. 也是解 ☆

全部解是 $-x+y+3 = C\cdot(x+y+1)^3$ $(C \in R)$ 和 $x+y+1=0$

④ $y' = \frac{x+2y+1}{2x+4y-1}$.

这是形如 $y' = f\left(\frac{ax+by+c}{dx+ey+f}\right)$ 且 $\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0$ 情形.

无法平移变齐次，直接换元 $u = x+2y$.

$\frac{du}{dx} = 1 + 2y'$ $\Rightarrow y' = \frac{u'-1}{2}$.

$\Rightarrow \quad u'-1 = 2 \cdot \frac{u+1}{2u-1} = \frac{2u+2}{2u-1} = 1 + \frac{3}{2u-1}$.

$\Rightarrow \quad u' = 1 + \frac{2u+2}{2u-1} = \frac{4u+1}{2u-1}$.

$\Rightarrow \quad \frac{2u-1}{4u+1}du = dx$.

$\Rightarrow \quad x + C = \int \frac{2u-1}{4u+1}du = \int \frac{1}{2} + \frac{-\frac{3}{2}}{4u+1}du$

$= \frac{u}{2} - \frac{3}{8}\ln|4u+1|$

$= \frac{x}{2} + y - \frac{3}{8}\ln|4x+8y+1|$.

$\Rightarrow \quad \ln|4x+8y+1| = \frac{8}{3}\left(y - \frac{x}{2}\right) + C$.

$\Rightarrow \quad 4x+8y+1 = Ce^{\frac{8}{3}(y-\frac{x}{2})} \quad (C \neq 0)$

检查 $4x+8y+1 = 0$ 也是解

$\Rightarrow$ 全部解是 $(4x+8y+1)^3 = Ce^{8y-4x} \quad (C \in \mathbb{R})$

⑤ $y'(x+2y) = 1$. 求 $y(0) = -1$ 的特解.

即 $y' = \frac{1}{x+2y}$ 形如 $y' = f(ax+by+c)$

换元 $u = x+2y$. $u' = 1+2y'$. $y' = \frac{u'-1}{2}$

$\Rightarrow \quad u'-1 = \frac{2}{u} \Rightarrow \frac{du}{dx} = 1 + \frac{2}{u} = \frac{u+2}{u}$

$\Rightarrow \quad dx = \frac{udu}{u+2} \Rightarrow$ ~~$\ln|x| + C$~~ $x + C = u - 2\ln|u+2|$.

$= x+2y - \ln(x+2y+2)^2$

$\Rightarrow \quad 2y = C + \ln(x+2y+2)^2$. $(C \in \mathbb{R})$.

代入 $(x,y) = (0,-1)$. : $-2 = C + \ln 0^2$ 不在该通解内!

检查过程. 唯一不充要的步骤是除以 $u+2$.

因为有可能 $u+2=0$!

$u+2=0$ 对应 $x+2y+2=0$. 即 $y=-1-\frac{x}{2}$.

检验: $x=0$ 时, $y=-1$

$y'=-\frac{1}{2}$. $x+2y=-2$ $\Rightarrow y'(x+2y)=1$ ✓

因此所求特解是 $x+2y+2=0$.

Rmk. 这称为奇解/包络. 图像上是通解曲线族的一条包络直线.

⑥ $x^2y'-\cos 2y=1$. $y(+\infty)=\frac{9}{4}\pi$.

解: $\frac{dx}{x^2}=\frac{dy}{1+\cos 2y}=\frac{1}{2}\sec^2 y\,dy$

$\Rightarrow -\frac{1}{x}=\frac{1}{2}\tan y+C$. 代入 $(x,y)=(+\infty,\frac{9}{4}\pi)$.

得: $0=\frac{1}{2}\tan\frac{9}{4}\pi+C$

$=\frac{1}{2}+C \Rightarrow C=-\frac{1}{2}$.

$\Rightarrow$ 所求为 $\tan y-1+\frac{2}{x}=0$.

即 $y=2\pi+\arctan\frac{x-2}{x}$ $(x>0)$.

Rmk. 所求通解是 $y=k\pi+\arctan\frac{Cx-2}{x}$ $(k\in\mathbb{Z}, C\in\mathbb{R})$. 这不违反一阶 ODE 通解仅含一个常数 C 的事实. 因为通积分为 $\tan y+\frac{2}{x}+C=0$ $(C\in\mathbb{R})$. k 的引入只是将该曲线分成不同的连续部分.

另外, 从该结构可见. $y(+\infty)=k\pi+\arctan C$

$y(-\infty)=k\pi-\arctan C$.

这些曲线被限制在 $y=(k\pm\frac{1}{2})\pi$ 之间, 这导致 $y=(k+\frac{1}{2})\pi$, $k\in\mathbb{Z}$ 都是奇解. 对应 C 取 $+\infty$.

⑦ $3y^2y' + 16x = 2xy^3$. $x \to +\infty$ 时，y 有界.

$\frac{3y^2dy}{y^3-8} = 2x\,dx$ 不充要 $\Rightarrow$ $y \equiv 2$ 是一个解

$\Rightarrow \ln|y^3-8| = C + x^2 \Rightarrow y = (8 + Ce^{x^2})^{\frac{1}{3}}$ $(C \neq 0)$

综合 $\Rightarrow$ $y = (8+Ce^{x^2})^{\frac{1}{3}}$ $(C \in \mathbb{R})$ 包含了 $y \equiv 2$.

欲有界 $\Leftrightarrow C = 0$. 故所求为 $y \equiv 2$.

⑧ (2024. 9.)

$f: \mathbb{R} \to \mathbb{R}$ 可导. $f'(x) = xf(x) + x\cdot\int_0^1 tf(t)\,dt$.

记 $\int_0^1 tf(t)\,dt = C$. 即 $f' = x(f+C)$

$\Rightarrow \frac{df}{f+C} = x\,dx \Rightarrow \ln(f(x)+C)^2 = x^2 + \tilde{C}$

$\Rightarrow f(x) = C_1 e^{\frac{x^2}{2}} - C_2$ 这里 $C_1 \neq 0$

$C_2 = \int_0^1 tf(t)\,dt$

会失去常值解.

(i) $\int_0^1 xf(x)\,dx = C_1\int_0^1 xe^{\frac{x^2}{2}}dx - C_2\int_0^1 x\,dx$

$= C_1(\sqrt{e}-1) - \frac{1}{2}C_2 = C_2$

$\Rightarrow C_2 = \frac{2}{3}C_1(\sqrt{e}-1)$.

$\Rightarrow f(x) = C\cdot\left(e^{\frac{x^2}{2}} - \frac{2}{3}(\sqrt{e}-1)\right)$. $(C \neq 0.)$

(ii) 检查常值解. $f(x) = -C = -\int_0^1 tf(t)\,dt$.

那么，$C = \int_0^1 tf(t)\,dt = -\int_0^1 Ct\,dt = -\frac{1}{2}C \Rightarrow C = 0$

综上：$f(x) = C\cdot\left(e^{\frac{x^2}{2}} - \frac{2}{3}(\sqrt{e}-1)\right)$. $(C \in \mathbb{R})$

2. 一元积分因子.

对形如 $f'(x)+g(x)f(x)=0$ 型方程. 同乘 $h(x)$

$$\Rightarrow \quad h(x)f'(x)+h(x)g(x)f(x)=0.$$

使左侧变成一个乘积型导数. $(h(x)f(x))'$

则 $(h(x)f(x))'=h(x)f'(x)+h'(x)f(x)$

$$=h(x)f'(x)+h(x)g(x)f(x)$$

要求满足 $h'(x)=h(x)g(x)$ 即 $(\ln h(x))'=g(x)$

故 $\ln h(x)=\int_a^x g(t)\,dt+C$

$$\Rightarrow \quad h(x)=Ce^{\int_a^x g(t)\,dt}.$$ 特别地, 取 $C=1$ 即可.

Eg. $f'(x)+f(x)\geq 0$. $f(0)\geq 0$. 则 $f(x)\geq 0$. $\forall x\geq 0$.

证明: $(e^x f(x))'\geq 0 \Rightarrow f(x)\geq e^{-x}f(0)\geq 0$. □

Eg. $f'(x)+\frac{1}{x}f(x)=0$.

$$h(x)=e^{\int_0^x \frac{1}{t}dt}=e^{\ln x}=x.$$

$$\Rightarrow \quad xf'(x)+f(x)=0 \quad \Rightarrow \quad (xf(x))'=0$$

$$\Rightarrow \quad xf(x)=C \quad \Rightarrow \quad f(x)=\frac{C}{x}.$$

Eg. $f'(x)+\frac{1}{1+x^2}f(x)=0$.

$$h(x)=e^{\int_0^x \frac{dt}{1+t^2}}=e^{\arctan x}$$

$$\Rightarrow \quad (e^{\arctan x}f(x))'=0 \quad \Rightarrow \quad f(x)=Ce^{-\arctan x}$$

[Gronwall 不等式].

$g(x) \geq 0$. $f(x) \leq c + \int_a^x f(t) g(t)\, dt$. 则 $f(x) \leq c e^{\int_a^x g(t)\, dt}$.

证明: 令 $F(x) = \int_a^x f(t) g(t)\, dt$.

$$F'(x) = f(x) g(x) \leq (c + F(x)) g(x)$$

F, F' 同时出现

=> $F'(x) - g(x) F(x) \leq c g(x)$

两乘积分因子. $\mu = e^{-\int_a^x g(t) dt}$

$$\Rightarrow \left(F(x) \cdot e^{-\int_a^x g(t) dt}\right)' \leq c g(x) e^{-\int_a^x g(t) dt}$$

$$= -c \cdot \left(e^{-\int_a^x g(t) dt}\right)'$$

[a, X] 上

积分: $F(x) \cdot e^{-\int_a^x g(t) dt} - \underbrace{F(a) \cdot e^0}_{F(a)=0} \leq -c \cdot \left(e^{-\int_a^x g(t) dt} - e^0\right)$.

$$\Rightarrow F(x) \leq e^{\int_a^x g(t) dt} \cdot c\left(1 - e^{-\int_a^x g(t) dt}\right)$$

$$= c \cdot \left(e^{\int_a^x g(t) dt} - 1\right).$$

$$\Rightarrow f(x) \leq c + F(x) \leq c \cdot e^{\int_a^x g(t) dt}. \quad \square.$$

应用举例①. Picard 定理唯一性部分.

如果 $\begin{cases} y' = f(x, y) \\ y(0) = y_0. \end{cases}$ 满足 $|f(x, y_1) - f(x, y_2)| \leq L \cdot |y_1 - y_2|$

则解是唯一的.

证明: 解满足 $y(x) = y_0 + \int_0^x f(t, y(t))\, dt$.

设有 2 个解 $y_1(x)$, $y_2(x)$.

$$\text{则 } |y_1(x) - y_2(x)| = \left| \int_0^x f(t, y_1(t)) - f(t, y_2(t))\, dt \right|$$

$$\leq \int_0^x |f(t, y_1(t)) - f(t, y_2(t))|\, dt$$

$$\leq L \cdot \int_0^x |y_1(t) - y_2(t)|\, dt.$$

这就是 Gronwall 不等式 $g(x) \equiv L \geq 0$. $c = 0$ 情形.

因此, $|y_1(x) - y_2(x)| \leq 0 \Rightarrow y_1(x) \equiv y_2(x)$. $\square$.

② 将一般 ODE 转化为恰当方程.

(i) $(x^2+y^2+x)\,dx + y\,dy = 0$.

同乘因子 $f(x)$. $\quad f(x)(x^2+y^2+x)\,dx + f(x)\,y\,dy = 0$

欲使之为恰当的，要求 $\frac{\partial}{\partial y}(f(x)(x^2+y^2+x)) = \frac{\partial}{\partial x}(f(x)\,y)$

即 $2y f(x) = f'(x)\,y$

即 $f'(x) - 2f(x) = 0$

$\Rightarrow f(x) = e^{2x}$.

(ii) $(3x^2y + 2xy + y^3)\,dx + (x^2+y^2)\,dy = 0$.

同乘 $f(x)$. 令 $\frac{\partial}{\partial y}(f(x)(3x^2y+2xy+y^3)) = \frac{\partial}{\partial x}(f(x)(x^2+y^2))$

即 $(3x^2+2x+3y^2) f(x) = 2x f(x) + x^2 f'(x) + y^2 f'(x)$

$\Leftrightarrow$ ~~$f'(x) - 3\left(1+\frac{y^2}{x^2}\right) f(x)$~~

$f'(x) - 3f(x) = 0 \quad \Rightarrow f(x) = e^{3x}$.

③ 积分因子的复杂选择.

$y^2(y\,dx - 2x\,dy) = x^3(x\,dy - 2y\,dx)$.

同乘 $x^a y^b$ 待定系数.

$(x^a y^{b+3} + 2x^{a+3} y^{b+1})\,dx - (2x^{a+1} y^{b+2} + x^{a+4} y^b)\,dy = 0$.

$\frac{\partial}{\partial y}(\cdot) = (b+3)x^a y^{b+2} + 2(b+1)x^{a+3} y^b$

$\frac{\partial}{\partial x}(\cdot) = -2(a+1)x^a y^{b+2} - (a+4)x^{a+3} y^b$.

令之相等: $\begin{cases} b+3 = -2(a+1) \\ 2(b+1) = -(a+4) \end{cases} \Leftrightarrow \begin{cases} a = -\frac{4}{3} \\ b = -\frac{7}{3} \end{cases}$

$\Rightarrow (x^{-\frac{4}{3}} y^{\frac{2}{3}} + 2x^{\frac{5}{3}} y^{-\frac{4}{3}})\,dx - (2x^{-\frac{1}{3}} y^{-\frac{1}{3}} + x^{\frac{8}{3}} y^{-\frac{7}{3}})\,dy = 0$.

积分. $-3x^{-\frac{1}{3}} y^{\frac{2}{3}} + \frac{3}{4} x^{\frac{8}{3}} y^{-\frac{4}{3}} = C$.